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Continuity of the spectrum on a class of upper triangular operator matrices [☆]

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ABSTRACT

Let $B(\mathcal{H})$ denote the algebra of operators on an infinite dimensional complex Hilbert space $\mathcal{H}$, and let $A^\circ \in B(\mathcal{K})$ denote the Berberian extension of an operator $A \in B(\mathcal{H})$. It is proved that the set theoretic function σ, the spectrum, is continuous on the set $\mathcal{C}(i) \subset B(\mathcal{H}_i)$ of operators A for which $\sigma(A) = \{0\}$ implies A is nilpotent (possibly, the 0 operator) and $A^\circ = \begin{pmatrix} \lambda & X \\ 0 & B \end{pmatrix} \begin{pmatrix} (A^\circ-\lambda)^{-1}(0) \\ \{(A^\circ-\lambda)^{-1}(0)\}^\perp \end{pmatrix}$ at every non-zero $\lambda \in \sigma_p(A^\circ)$ for some operators X and B such that $\lambda \notin \sigma_p(B)$ and $\sigma(A^\circ) = \{\lambda\} \cup \sigma(B)$. If $\mathcal{C}_S(m)$ denotes the set of upper triangular operator matrices $A = (A_{ij})_{i,j=1}^m \in B(\bigoplus_{i=1}^n \mathcal{H}_i)$, where $A_{ii} \in \mathcal{C}(i)$ and A_{ii} has SVEP for all $1 \leqslant i \leqslant m$, then σ is continuous on $\mathcal{C}_S(m)$. It is observed that a considerably large number of the more commonly considered classes of Hilbert space operators constitute sets $\mathcal{C}(i)$ and have SVEP.

1. Introduction

Let $\mathcal{H}$ (similarly, $\mathcal{H}_i$ for $i = 1, 2, \ldots, m$ for some natural number m) denote a complex infinite dimensional Hilbert space, and let $B(\mathcal{H})$ denote the algebra of operators (equivalently, bounded linear transformations) on $\mathcal{H}$. Let $\mathcal{K}$ denote the set of all compact subsets of the complex plane **C**. Equipping $\mathcal{K}$ with the Hausdorff metric, one may consider the "spectrum" σ as a function $\sigma : B(\mathcal{H}) \to \mathcal{K}$ mapping operators $T \in B(\mathcal{H})$ into their spectrum $\sigma(T)$. The function σ is upper semi-continuous, but has points of discontinuity [12, p. 56]. Newburgh, in his seminal paper [17], studied spectral continuity in the setting of general Banach algebras; a detailed study in the case in which the Banach algebra is a C^*-algebra of operators acting on a complex separable Hilbert space has been carried out by Conway and Morrel [3,4]. Studies identifying sets $\mathcal{C}$ of operators for which σ becomes continuous when restricted to $\mathcal{C}$ has been carried out by a number authors (see, for example [5,7, 13,15,16]). The present paper derives its inspiration from [15], where Kim and Lee prove that the spectrum is continuous on the set of *quasi-m-hyponormal operators*. Here a quasi-m-hyponormal operator is an upper triangular operator matrix $A = (A_{ij})_{i,j=1}^m$, $A_{ij} = 0$ for all $i > j$, such that the entries A_{ii} along the principal diagonal of the matrix are hyponormal operators (on a separable infinite dimensional complex Hilbert space). This paper considers spectral continuity on the set $\mathcal{C}_S(m)$ of upper triangular operator matrices $A = (A_{ij})_{i,j=1}^m \in B(\bigoplus_{i=1}^n \mathcal{H}_i)$ such that the elements $A_{ii} \in B(\mathcal{H}_i)$ have the single-valued extension property and belong to a set $\mathcal{C}(i)$, defined as follows.

The Berberian extension theorem [2] says that given an operator $T \in B(\mathcal{H}_i)$ there exists a Hilbert space $\mathcal{K}_i \supseteq \mathcal{H}_i$ and an isometric $*$-isomorphism $T \to T^\circ \in B(\mathcal{K}_i)$ preserving order such that $\sigma(T) = \sigma(T^\circ)$ and $\sigma_p(T^\circ) = \sigma_a(T^\circ) = \sigma_a(T)$. Here

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σ_p and σ_a denote, respectively, the point spectrum and the approximate point spectrum. Let $T-\lambda = T-\lambda I$. The set $\mathcal{C}(i)$ consists of (all) the operators $T \in B(\mathcal{H}_i)$ for which $\sigma(T)=\{0\}$ implies T is nilpotent (possibly, the 0 operator) and T° satisfies the property:

$$T^\circ = \begin{pmatrix} \lambda & X \\ 0 & B \end{pmatrix} \begin{pmatrix} (T^\circ-\lambda)^{-1}(0) \\ \{(T^\circ-\lambda)^{-1}(0)\}^\perp \end{pmatrix}, \tag{1}$$

at every non-zero $\lambda \in \sigma_p(T^\circ)$ for some operators X and B such that $\lambda \notin \sigma_p(B)$ and $\sigma(T^\circ) = \{\lambda\} \cup \sigma(B)$. We prove:

Theorem 1.1. *σ is continuous on $\mathcal{C}(i)$.*

Recall that an operator $A \in B(\mathcal{H})$ has the single-valued extension property, SVEP for short, at a point $\lambda \in \mathbf{C}$ if for every disc $\mathcal{D}$ centered at λ the only analytic function $f : \mathcal{D} \to \mathcal{H}$ satisfying $(A-\mu)f(\mu) = 0$ for all $\mu \in \mathcal{D}$ is the function $f \equiv 0$. Every operator A has SVEP on its resolvent set and at points in $\operatorname{iso}\sigma(A)$, the isolated points of the spectrum of A. We say that A has SVEP if it has SVEP at every point in $\sigma(A)$. We prove:

Theorem 1.2. *σ is continuous on $\mathcal{C}_S(m)$.*

We remark here that our definition of the sets $\mathcal{C}(i)$ is not as unnatural as it may seem at first sight. A substantially large number of the commonly considered classes of Hilbert space operators are characterised by a positivity condition which has a natural Berberian extension interpretation. This is, in particular, true of the following classes of operators (see also [11] and [9]).

An operator $T \in B(\mathcal{H})$ is *p-hyponormal*, $0 < p \leqslant 1$, if $|T^*|^{2p} \leqslant |T|^{2p}$; *$w$-hyponormal* if $|\tilde{T}^*| \leqslant |T| \leqslant |\tilde{T}|$, where, for the polar decomposition $T = U|T|$ of T, $\tilde{T}$ is the Aluthge transform $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ of T; *M-hyponormal* if there exists a number $M \geqslant 1$ such that $|T^* - \bar{\lambda}|^2 \leqslant M|T-\lambda|^2$ for all complex λ; *(q,p)-quasihyponormal* for some positive integer q and $0 < p \leqslant 1$ if $T^{*q}(|T|^{2p} - |T^*|^{2p})T^q \geqslant 0$; *of class $\mathcal{A}$* if $|T|^2 \leqslant |T^2|$; *quasiclass $\mathcal{A}$* if $T^*(|T^2| - |T|^2)T \geqslant 0$; *$*$-paranormal* if $\|T^*x\|^2 \leqslant \|T^2x\|$ for every unit vector $x \in \mathcal{H}$, and T is *paranormal* if $\|Tx\|^2 \leqslant \|T^2x\|$ for every unit vector $x \in \mathcal{H}$.

The following inclusions are known to be proper: hyponormal $\subset$ p-hyponormal $\subset$ w-hyponormal $\subset$ class $\mathcal{A}$ $\subset$ paranormal (see [9, p. 144] for an appropriate reference). Observe that a $(q,1)$-quasihyponormal operator is q-quasihyponormal. We note that operators belonging to these classes of operators have SVEP; see [10], Theorem 2.8, Example 2.1 and Example 2.3. Also, if T is a quasinilpotent operator in any one of these classes, then T is nilpotent (possibly, the 0 operator). For example, if $\sigma(T) = \{0\}$ and T is: (a) (q,p)-quasihyponormal or quasiclass $\mathcal{A}$, then T is nilpotent [14,19]; (b) M-hyponormal, then $T = 0$ [18]; (c) $*$-paranormal or paranormal (hence, normaloid), then (again) $T = 0$.

It is evident from the definition of the classes above that if T is an operator in any one of these classes, then its Berberian extension T° is in the same class (i.e., these classes are stable under the operation of taking Berberian extension). It is well known that the eigenvalues of M-hyponormal operators are normal (i.e., the eigenspaces corresponding to the eigenvalues are reducing), and that the non-zero eigenvalues of (q,p)-quasihyponormal, quasiclass $\mathcal{A}$ and $*$-paranormal operators are normal (see [1,19,14]). Furthermore, if T is paranormal and $(0 \neq)\lambda \in \sigma_p(T)$, then $T = \begin{pmatrix} \lambda & X \\ 0 & B \end{pmatrix} \begin{pmatrix} (T-\lambda)^{-1}(0) \\ \{(T-\lambda)^{-1}(0)\}^\perp \end{pmatrix}$, where B is paranormal (implies $\sigma(T) = \{\lambda\} \cup \sigma(B)$) and $\lambda \notin \sigma_p(B)$ [20]. Putting all this together, it follows that if T is in any one of the classes defined above, then $T \in \mathcal{C}(i)$ and T has SVEP.

The set $\mathcal{C}_S(m)$ is large: it contains, amongst others, the set of upper triangular operator matrices $A = (A_{ij})_{i,j=1}^m \in B(\bigoplus_{i=1}^n \mathcal{H}_i)$ such that A_{ii} is hyponormal or p-hyponormal or M-hyponormal or k-quasihyponormal or class $\mathcal{A}$ or quasiclass $\mathcal{A}$ or paranormal for all $1 \leqslant i \leqslant m$. The following corollary, which generalises [15, Theorem 1] (and a number of other extant results), is immediate from Theorem 1.2.

Corollary 1.3. *If $\mathcal{C}(m)$ denotes the set of upper triangular operators $A = (A_{ij})_{i,j=1}^m \in B(\bigoplus_{i=1}^n \mathcal{H}_i)$, where $A_{ii} \in B(\mathcal{H}_i)$ is in one of the classes defined above for all $1 \leqslant i \leqslant m$, then σ is continuous on $\mathcal{C}(m)$.*

2. Proof of Theorems 1.1 and 1.2

Given an operator $T \in B(\mathcal{H})$, let $\alpha(T) = \dim(T^{-1}(0))$ and $\beta(T) = \dim(\mathcal{H} \setminus T(\mathcal{H}))$. T is upper semi-Fredholm if $T(\mathcal{H})$ is closed and $\alpha(T) < \infty$, and then the index of T, $\operatorname{ind}(T)$, is defined by $\operatorname{ind}(T) = \alpha(T) - \beta(T)$. T is said to be Fredholm if $T(\mathcal{H})$ is closed and the deficiency indices $\alpha(T)$ and $\beta(T)$ are (both) finite. In the following, we shall denote the set of accumulation points of $\sigma(T)$ by $\operatorname{acc}\sigma(T)$, and the statement "$\lambda \in \liminf_n S_n$", S_n compact for all n, shall mean that for every open $U \ni \lambda$, $S_n \cap U \neq \emptyset$ for all sufficiently large n.

Proof of Theorem 1.1. Start by recalling that the function σ is upper semi-continuous, and that if $\{A_n\} \subset B(\mathcal{H}_i)$ is a sequence which converges in the operator norm topology to $A \in B(\mathcal{H}_i)$ then $\liminf_n \sigma(A_n) \subset \sigma(A)$ [12]. Thus to prove the theorem it would suffice to prove that if $\{A_n\} \subset \mathcal{C}(i)$ is a sequence of operators such that $\lim_{n\to\infty} \|A_n - A\| = 0$ for some operator

$A \in \mathcal{C}(i)$, then $\sigma(A) \subset \liminf_n \sigma(A_n)$. The hypothesis "$T$ quasinilpotent implies T nilpotent for every $T \in \mathcal{C}(i)$" implies that if $\sigma(A_n) = \{0\}$ for all n for a sequence $\{A_n\} \subset \mathcal{C}(i)$ converging in the uniform topology to $A \in \mathcal{C}(i)$, then $\sigma(A) = \{0\}$. Hence we may assume in the following that $\sigma(A_n)$, all n, does not consist of the singleton set $\{0\}$.

Observe that if $\lambda \in \operatorname{iso}\sigma(A)$, then $\lambda \in \liminf_n \sigma(A_n)$. This follows from an argument of Newburgh [17]: indeed, if $\lambda \in \operatorname{iso}\sigma(A)$, then for every neighbourhood $\mathcal{N}(\lambda)$ of λ there exists a positive integer N such that $\sigma(A_n) \cap \mathcal{N}(\lambda) \neq \emptyset$ for all $n > N$. This implies that to prove the theorem it would suffice to prove that $\lambda \in \operatorname{acc}\sigma(A) \Longrightarrow \lambda \in \liminf_n \sigma(A_n)$.

Let, as before, $T^\circ \in B(\mathcal{K})$ denote the Berberian extension of an operator in $T \in B(\mathcal{H})$. Then, with A and the sequence $\{A_n\}$ as above, we have that

$$\sigma(A) = \sigma\big(A^\circ\big), \qquad \sigma(A_n) = \sigma\big(A_n^\circ\big) \quad \text{and} \quad \sigma_a(A) = \sigma_a\big(A^\circ\big) = \sigma_p\big(A^\circ\big).$$

Consequently, one has that

$$\operatorname{acc}\sigma(A) \subset \liminf_n \sigma(A_n) \quad \Longleftrightarrow \quad \operatorname{acc}\sigma\big(A^\circ\big) \subset \liminf_n \sigma\big(A_n^\circ\big).$$

We divide the proof into three steps. Step 1 below reduces the problem to that of considering points in $\operatorname{acc}\sigma_a(A^\circ)$.

Step 1: $\lambda \in \operatorname{acc}\sigma(A^\circ)$ and $\lambda \notin \sigma_p(A^\circ) \Longrightarrow \lambda \in \liminf_n \sigma(A_n^\circ)$. If $\lambda \in \operatorname{acc}\sigma(A^\circ)$ and $\lambda \notin \sigma_p(A^\circ)$, then $A^\circ - \lambda$ is left invertible, hence upper semi-Fredholm with $\alpha(A^\circ - \lambda) = 0$. Suppose that $\lambda \notin \liminf_n \sigma(A_n^\circ)$. Then there exists a $\delta > 0$, a neighbourhood $\mathcal{N}_\delta(\lambda)$ of λ and a subsequence $\{A_{n_k}^\circ\}$ of $\{A_n^\circ\}$ such that $\sigma(A_{n_k}^\circ) \cap \mathcal{N}_\delta(\lambda) = \emptyset$ for every $k \geqslant 1$. This implies that $A_{n_k}^\circ - \mu$ is Fredholm and $\operatorname{ind}(A_{n_k}^\circ - \mu) = 0$ for every $\mu \in \mathcal{N}_\delta(\lambda)$. Thus, since $\|(A_{n_k}^\circ - \lambda) - (A^\circ - \lambda)\| \to 0$ as $n \to 0$, the continuity of the index implies that $\operatorname{ind}(A^\circ - \lambda) = 0$ ($\Longrightarrow A^\circ - \lambda$ is Fredholm). Since $\alpha(A^\circ - \lambda) = 0$, it follows that $\lambda \notin \sigma(A^\circ)$ – a contradiction. Hence $\lambda \in \liminf_n \sigma(A_n^\circ)$.

Observe that if an operator $A^\circ \in B(\mathcal{K}_i)$ has a representation of type (1) at every $(0 \neq)\lambda \in \sigma_p(A^\circ)$, where $\lambda \notin \sigma_p(B)$ and $\sigma(A^\circ) = \{\lambda\} \cup \sigma(B)$, then $\sigma_a(B) = \sigma_p(B) = \sigma_p(A^\circ) \setminus \{\lambda\}$. We shall use this observation in our proof of Steps 2 and 3. Throughout the following, the neighbourhood $\mathcal{N}_\delta(\lambda)$ of λ and the subsequence $\{A_{n_k}^\circ\}$ of $\{A_n^\circ\}$ will be defined as in Step 1.

Step 2: $0 \neq \lambda \in \operatorname{acc}\sigma_a(A^\circ) \Longrightarrow \lambda \in \liminf_n \sigma(A_n^\circ)$. If $0 \neq \lambda \in \sigma_a(A^\circ) = \sigma_p(A^\circ)$, then

$$A^\circ = \begin{pmatrix} \lambda & X \\ 0 & B \end{pmatrix} \begin{pmatrix} (A^\circ - \lambda)^{-1}(0) \\ \{(A^\circ - \lambda)^{-1}(0)\}^\perp \end{pmatrix},$$

where $\sigma_a(B) = \sigma_p(B) = \sigma_p(A^\circ) \setminus \{\lambda\}$. Evidently, $B - \lambda \in B(\{(A^\circ - \lambda)^{-1}(0)\}^\perp)$ is upper semi-Fredholm and $\alpha(B - \lambda) = 0$. There exists an $\epsilon > 0$ such that $B - (\lambda - \mu_o)$ is upper semi-Fredholm (with $\operatorname{ind}(B - (\lambda - \mu_o)) = \operatorname{ind}(B - \lambda)$ and $\alpha(B - (\lambda - \mu_o)) = 0$) for every μ_o satisfying $0 < |\mu_o| < \epsilon$. Choose $0 < \epsilon < \delta$ and set $\mu = \lambda - \mu_o$ $(0 < |\mu_o| < \epsilon)$. (Here $\delta > 0$ is as in Step 1.) Then $B - \mu$ is upper semi-Fredholm, $\operatorname{ind}(B - \mu) = \operatorname{ind}(B - \lambda)$ and $\alpha(B - \mu) = 0$. This implies that

$$A^\circ - \mu = \begin{pmatrix} \lambda - \mu & X \\ 0 & B - \mu \end{pmatrix}$$

is upper semi-Fredholm,

$$\operatorname{ind}\big(A^\circ - \mu\big) = \operatorname{ind}\left[\begin{pmatrix} 1 & 0 \\ 0 & B - \mu \end{pmatrix} \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda - \mu & 0 \\ 0 & 1 \end{pmatrix}\right] = \operatorname{ind}(B - \mu)$$

and $\alpha(A^\circ - \mu) = 0$. Evidently, $A_{n_k}^\circ - \mu$ is Fredholm, with $\operatorname{ind}(A_{n_k}^\circ - \mu) = 0$, and $\lim_{n\to\infty} \|(A_{n_k}^\circ - \mu) - (A^\circ - \mu)\| = 0$. It follows from the continuity of the index that $\operatorname{ind}(A^\circ - \mu) = 0$ and $A^\circ - \mu$ is Fredholm. Since $\alpha(A^\circ - \mu) = 0$, $\mu \notin \sigma(A^\circ)$ for every μ in a deleted ϵ-neighbourhood of λ. Since this contradicts our hypothesis that $\lambda \in \operatorname{acc}\sigma_p(A^\circ)$, we must have that $\lambda \in \liminf_n \sigma(A_n^\circ)$.

Step 3: $\lambda = 0 \in \operatorname{acc}\sigma_p(A^\circ) \Longrightarrow 0 \in \liminf_n \sigma(A_n^\circ)$. Assume that $0 \in \operatorname{acc}\sigma_p(A^\circ)$ and $0 \notin \liminf_n \sigma(A_n^\circ)$. Then every neighbourhood $\mathcal{N}_\epsilon(0)$ of 0, $0 < \epsilon < \delta$, contains a non-zero sequence $\{\mu_m\} \subset \sigma_p(A^\circ)$ such that $\lim_{m\to\infty} \mu_m = 0$ and $\mu_m \notin \sigma(A_{n_k}^\circ)$ for all integers $m, k \geqslant 1$. Choose one such neighbourhood $\mathcal{N}_\epsilon(0)$ of 0 and a $0 \neq \mu_o \in \{\mu_m\}$. Then

$$A^\circ = \begin{pmatrix} \mu_o & X \\ 0 & B \end{pmatrix} \begin{pmatrix} (A^\circ - \mu_o)^{-1}(0) \\ \{(A^\circ - \mu_o)^{-1}(0)\}^\perp \end{pmatrix},$$

where $\mu_o \notin \sigma_p(B)$. Arguing as above (this time with λ replaced by μ_o) it is seen that $\mu_o \notin \sigma(A^\circ)$. This is a contradiction. Hence $0 \in \liminf_n \sigma(A_n^\circ)$. $\square$

Proof of Theorem 1.2. Let $A = (A_{ij})_{i,j=1}^m \in \mathcal{C}_S(m)$, and let $A_n = (A_{n(ij)})_{i,j=1}^m \in \mathcal{C}_S(m)$ be such that $\lim_{n\to\infty} \|A_n - A\| = 0$. Then

$$\|A_{n(ii)} - A_{ii}\| \leqslant \|A_n - A\| \to 0 \quad \text{as } n \to \infty,$$

for all $1 \leqslant i \leqslant m$. The SVEP hypothesis on A_{ii} and $A_{n(ii)}$, $1 \leqslant i \leqslant m$, implies that

$$\sigma(A) = \bigcup_{i=1}^{m} \sigma(A_{ii}) \quad \text{and} \quad \sigma(A_n) = \bigcup_{i=1}^{m} \sigma(A_{n(ii)})$$

[8, Lemma 3.2]. Observe that if $\lambda \in \sigma(A)$, then $\lambda \in \sigma(A_{ii})$ for at least one value of i, say $i = k$. Hence, since $\lambda \in \sigma(A_{kk})$ implies $\lambda \in \liminf_n \sigma(A_{n(kk)})$ (by Theorem 1.1), $\lambda \in \liminf_n \sigma(A_n)$. $\square$

Remark 2.1. The hypothesis that A_{ii} has SVEP for all $1 \leqslant i \leqslant m$ is not essential to the validity of Theorem 1.2. Indeed, if we let $\mathcal{C}(m)$ denote the set of upper triangular operators $A = (A_{ij})_{i,j=1}^{m} \in B(\bigoplus_{i=1}^{n} \mathcal{H}_i)$ such that $A_{ii} \in \mathcal{C}(i)$ for all $1 \leqslant i \leqslant m$, then the hypothesis $\sigma(A) = \bigcup_{i=1}^{m} \sigma(A_{ii})$ for all $A \in \mathcal{C}(m)$ would do. It is apparent from the proof of Theorem 1.1 that σ_a is continuous on $\mathcal{C}(i)$. Hence, *if $\sigma_a(A) = \bigcup_{i=1}^{m} \sigma_a(A_{ii})$ for all $A \in \mathcal{C}(m)$, then σ_a is continuous on $\mathcal{C}(m)$.*

Remark 2.2. The ascent $\mathrm{asc}(T)$ (resp., descent $\mathrm{dsc}(T)$) of an operator $T \in B(\mathcal{H})$ is the least non-negative integer d such that $T^{-d}(0) = T^{-(d+1)}(0)$ (resp., $T^d(\mathcal{H}) = T^{d+1}(\mathcal{H})$). The Browder spectrum σ_b and the Weyl spectrum σ_w of T are the sets $\sigma_b(T) = \{\lambda \in \sigma(T)\colon\ T - \lambda$ is not Fredholm or one of $\mathrm{asc}(T - \lambda)$ and $\mathrm{dsc}(T - \lambda)$ is infinite$\}$ and $\sigma_w(T) = \{\lambda \in \sigma(T)\colon\ T - \lambda$ is not Fredholm or $\mathrm{ind}(T - \lambda) \neq 0\}$. Evidently, $\sigma_w(T) \subseteq \sigma_b(T)$: a necessary and sufficient condition for $\sigma_b(T) = \sigma_w(T)$ is that T has SVEP on $\sigma(T) \setminus \sigma_w(T)$. For operators $A \in \mathcal{C}_S(m)$, $\sigma_b(A) = \bigcup_{i=1}^{m} \sigma_b(A_{ii}) = \bigcup_{i=1}^{m} \sigma_w(A_{ii}) = \sigma_w(A)$ [8, Proposition 3.5(iii)], and we have the following.

Corollary 2.3. *σ_b and σ_w are continuous on $\mathcal{C}_S(m)$.*

Proof. If $\sigma_b(T) = \sigma_w(T)$ for a Hilbert space operator T, then the statements (i) σ is continuous at T, (ii) σ_b is continuous at T, and (iii) σ_w is continuous at T are equivalent [6, Theorem 2.2]. $\square$

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